10th International Conference on Hydroinformatics
HIC 2012, Hamburg, GERMANY

ON THE REPRESENTATION OF BOTTOM SHEAR STRESS IN Z-LAYER MODELS

FRANK W. PLATZEK (1,2,3), GUUS S. STELLING (1,2), JACEK A. JANKOWSKI (3) AND REGINA PATZWAHL (3)
(1): Fluid Mechanics Section, Faculty of Civil Engineering and Geosciences, Delft University of Technology, Stevinweg 1, 2628 CN Delft, the Netherlands
(2): Deltares, Rotterdamseweg 185, 2629 HD Delft, the Netherlands
(3): Department of Hydraulic Engineering in Inland Areas, Federal Waterways Engineering and Research Institute (BAW), Kußmaulstraße 17, 76187, Karlsruhe, Germany

Bottom friction plays an important role in modelling river flows. In three-dimensional (3D) models, the vertical discretization is commonly based on σ-layers or z-layers. In this paper we focus on a well-known problem encountered when applying z-layers: local truncation errors in the computation of bottom shear stress and near-bed turbulence along a sloping bottom as e.g. in the case of 3D river simulations. This problem stems from the 'staircase' representation of the bottom and results in difficulties in the computation of morphological changes. We consider uniform channel flow and analyze the influence of variations in near-bed layer thickness on the local truncation errors in the vertical diffusion term. Application of both an algebraic turbulence model, based on a prescribed mixing-length and the standard k-ε turbulence model to compute the eddy viscosity is investigated. We consider two approaches that reduce the local truncation errors and inspect their applicability for more general flow situations.

INTRODUCTION

The computation of bottom shear stress and near-bed turbulence in three-dimensional (3D) hydrodynamic models is of key importance for determining flow resistance and morpho-dynamics. For the vertical discretization in such 3D models, commonly either terrain-following σ-layers (Phillips [10]) or strictly horizontal z-layers are used (Figure 1). Both approaches have their advantages and disadvantages.

Using σ-layer models, problems arise when modelling stratified flow above steep bottom slopes, as shown by e.g. Stelling and Van Kester [12] and Van Kester et al. [5]. The z-layer discretization, on the other hand, results in a 'staircase' representation of the bottom and free surface, see Figure 1. Even using a partial-cell approach (Adcroft et al. [1], Pacanowski and Gnanadesikan [8]), the staircase boundaries cause problems. Firstly, implicit form drag may be generated due to inadequate treatment of the advection terms (see e.g. Beckmann and Döscher [3], Chen [4], Penduff et al. [9], Zhao [14] and Kleptsova et al. [6]). Secondly, large ratios in layer thickness - occurring when the bottom or free surface passes through a layer interface - result in discontinuities in the velocity and shear stress (Figure 1). This is especially critical for the bottom shear stress distribution as it is often applied in (coupled) sediment transport and morphodynamic models.

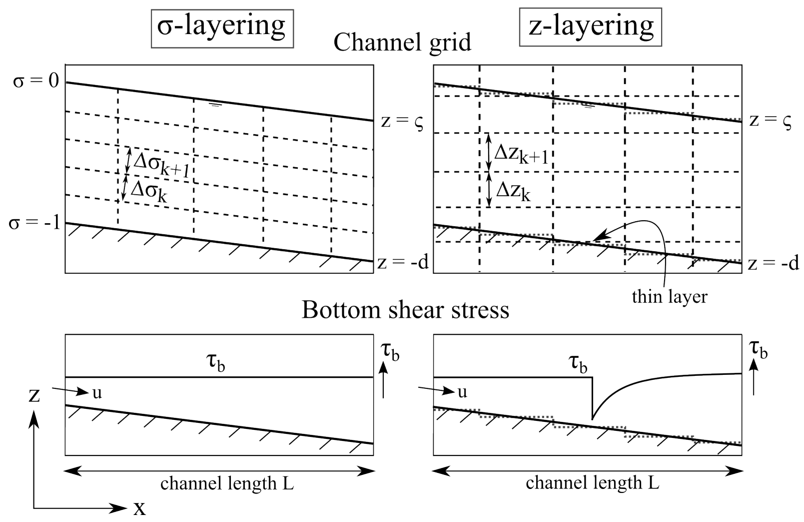

Figure 1. Vertical grid structure and bottom shear stress for uniform channel flow, using the σ-layer grid (left) and the z-layer grid (right).

Both of these errors diminish with increasing grid resolution. However, e.g. Winton et al [13] showed that the required horizontal and vertical resolutions would severely limit the applicability of a model to large-scale, long-term simulations. Most of the applications found in the mentioned literature concern ocean modelling, focusing on the influence of a step-like bottom on free-surface wave dispersion or spurious diffusion of density currents.

In this paper we aim to reduce the problem of representing bed shear stress along a bottom slope in *z*-layer models. For this purpose, we consider steady uniform flow along a constantly sloping channel (Figure 1). We inspect the local truncation errors with respect to the analytical solution, for the situation where we have a large near-bed layer thickness ratio, due to the partial-cell approach. We consider one existing method to reduce the local truncation errors and propose a new one. Only the latter is found suitable for more general flow situations and combination with the k-ε turbulence model. We present results obtained with the new approach and compare with a conventional method and the analytical solution. Finally, we provide conclusions and an outlook to future research.

FLOW MODEL

For uniform channel flow along a constant (mildly-)sloping bottom, the horizontal velocity u and vertical turbulent eddy viscosity ν_τ only vary with the depth. Uniform channel flow forms due the balance of frictional and gravitational forces. The averaged flow field shows a logarithmic u-profile and a parabolic ν_τ-profile. For our study, it is sufficient to investigate the vertical diffusion term only (omitting advection, horizontal diffusion, lateral effects and the transient term). Specifying $\tilde{p}$ as the kinematic pressure (pressure scaled with density ρ), the simplified horizontal momentum equation reads:

$$\tilde{p}_x = p_x / \rho = (\nu_\tau u_z)_z \tag{1}$$

The pressure gradient in horizontal direction $\tilde{p}_x$ is assumed to be equal to the free-surface slope, multiplied by the gravitational acceleration g and the free-surface slope is assumed parallel to the bottom, i.e. $\tilde{p}_x = g\zeta_x = -gi_b$, where i_b is the bottom slope (positive downwards).

The water column is discretized using horizontal z-layers. We improve the staircase representation of the bottom and free-surface level by applying a partial-cell approach, resulting in a varying layer thickness near the bottom and free-surface. At the free-surface and bottom we specify the shear stress (wind and bottom friction) as boundary conditions.

Discretizing the diffusion term using central differences one obtains for a layer k:

$$-gi_b = \frac{\nu_{\tau,k+\frac{1}{2}} \dfrac{u_{k+1} - u_k}{\Delta z_{k+\frac{1}{2}}} - \nu_{\tau,k-\frac{1}{2}} \dfrac{u_k - u_{k-1}}{\Delta z_{k-\frac{1}{2}}}}{\Delta z_k} \tag{2}$$

where Δz_k is the layer thickness of layer k and $\Delta z_{k+1/2} = \frac{1}{2}(\Delta z_k + \Delta z_{k+1})$. The viscosity ν_τ is given (or computed) at the layer interfaces. At this point we are merely interested in the effects of large ratios in layer thickness near the bottom. We therefore only consider the equation in the layer containing the bottom: $k = m$. In Eq. (2) we insert the boundary condition for the bottom shear stress $\tau_b / \rho = \nu_\tau u_z = u_*^2$, where u_*^2 is the shear velocity and we introduce $R_m = \Delta z_{m+1} / \Delta z_m$ as the layer thickness ratio between layers $m+1$ and m.

The equation for layer m becomes:

$$-gi_b = \nu_{\tau,m+\frac{1}{2}} \frac{u_{m+1} - u_m}{\left(\frac{1}{2} + \frac{1}{2} R_m\right)\Delta z_m^2} - \frac{u_*^2}{\Delta z_m} \tag{3}$$

ANALYSIS OF THE EQUATIONS

Discontinuities in velocities and bottom shear stress are known to occur when the bottom passes through a layer interface, introducing thin layers. We therefore investigate the behaviour of the local truncation error of the diffusion term as a function of the ratio in near-bed layer thickness R_m.

The analytical expressions for the velocity and turbulent eddy viscosity are:

$$u(z) = \frac{u_*}{\kappa} \ln\left(\frac{z + z_0}{z_0}\right) \tag{4}$$

$$\nu_\tau(z) = \kappa u_*(z + z_0)\left(1 - \frac{z}{H}\right) \tag{5}$$

where H is the total water depth, z_0 is the roughness height and κ the von Kármán constant. The latter expression for ν_τ corresponds to the application of an algebraic turbulence model based on the mixing-length concept (see e.g. Nezu and Nakagawa [7]). We assume these relations hold at least (approximately) in the near-bed layers. In the tests (Figures 4 and 5) we investigate the local truncation errors when the standard k-ε turbulence model is applied and compare with these analytical profiles.

Substituting expressions (4) and (5) in Eq. (3) yields the local truncation error in the near-bed layer e_m:

$$e_m = \frac{u_*^2}{H}\left[\frac{2(\Delta z_m - H)}{\Delta z_m(1 + R_m)}\ln\left(\frac{z_0 + \Delta z_m\left(1 + \frac{1}{2}R_m\right)}{z_0 + \frac{1}{2}\Delta z_m}\right) + \frac{H}{\Delta z_m}\right] - gi_b \tag{6}$$

If one integrates Eq. (2) over all layers, one obtains the result $gi_b = u_*^2 / H$ (assuming zero wind shear stress). Using this expression we obtain the relative local truncation error in the near-bed layer $E_m = e_m / (gi_b)$:

$$E_m = \frac{2(\Delta z_m - H)}{\Delta z_m(1 + R_m)}\ln\left(\frac{z_0 + \Delta z_m\left(1 + \frac{1}{2}R_m\right)}{z_0 + \frac{1}{2}\Delta z_m}\right) + \frac{H}{\Delta z_m} - 1 \tag{7}$$

This term depends on R_m, but also on H, z_0 and Δz_m. We can reduce the number of variables and gain some more insight in the error by introducing the ratios $R_0 = z_0 / \Delta z_m$ and $R_H = H / \Delta z_m$. Inserting these expressions in Eq. (7), we obtain:

$$E_m = \frac{2(1 - R_H)}{1 + R_m}\ln\left(\frac{R_0 + 1 + \frac{1}{2}R_m}{R_0 + \frac{1}{2}}\right) + R_H - 1 \tag{8}$$

R_0 is usually much smaller than the other ratios and can commonly be neglected. We therefore focus on the behaviour of the truncation error as a function of R_m and R_H. Figure 2 displays E_m (in %, i.e. multiplied by 100) as a function of these two ratios for a roughness height $z_0 = 0.0023$m. Three things can be noted:

- The error $E_m < 20\%$ for $R_m \approx 1$ (equidistant near-bed layering) for $R_H <$ 10-15.
- The error $E_m < 20\%$ for $R_H \approx 3$ ($H \approx 3\Delta z_m$) for $R_m <$ 6-8.
- The error grows rapidly in all other situations ($E_m > 500\%$).

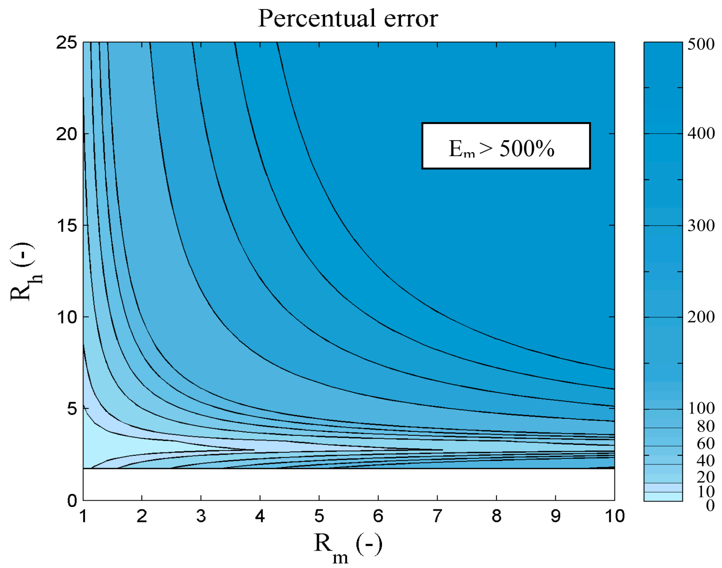

Figure 2. The relative local truncation error E_m of the vertical diffusion term (w.r.t. the analytical solution for uniform channel flow, using the algebraic turbulence model) as a function of the ratios R_m and R_H (using z_0 = 0.0023m).

IMPROVEMENTS

In this section, we investigate two possible approaches to reduce the local truncation errors in the approximation of the velocity gradients near the bottom. Note that one usually can not influence the value of R_H. We therefore do not consider it an option to aim for $R_H \approx 3$.

Existing approach: logarithmic transformation or discretization

A logical approach is to apply a logarithmic coordinate transformation (see e.g. Zijlema [14]) or a logarithmic discretization (e.g. Arya [2]). Others use the degrees of freedom in the numerical approximation of the vertical diffusion term by polynomial fitting, to achieve both consistency and a zero truncation error, also for non-equidistant layering (see e.g. Stelling [10]). These approaches yield accurate results for flows that are very close to uniform channel flow, also for moderate ratios in near-bed layer thickness. However, already when the velocity profiles are slightly non-logarithmic, the results strongly deteriorate, especially when combined with the k-ε turbulence model. As velocity profiles are rarely exactly logarithmic, we consider an approach that also works with the application of the k-ε turbulence model.

New approach: modification of the near-bed layer distribution

The local truncation errors were found to be smallest, for R_m = 1, i.e. for equidistant near-bed layering (for moderate ratios R_H). We therefore investigate the use of a remapping to an equidistant layer distribution near the bed (see Figure 3).

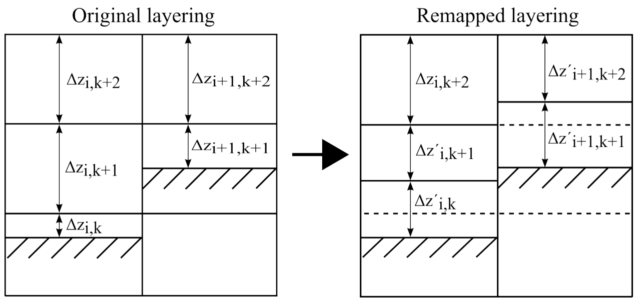

Figure 3. Remapping of the two near-bed layers to ensure a locally-equidistant layering.

RESULTS

We model uniform channel flow along a 1000m channel, using 25 cells horizontally and 10 vertical layers. The bottom crosses a layer interface once, at x = 500m, introducing a near-bed layer ratio $R_m \approx 250$. We apply the partial-cell approach, a z_0 of 0.0023m and the k-ε turbulence model (the algebraic turbulence model yields very similar results). Results obtained without near-bed layer modification are compared to results obtained applying the proposed mapping to an equidistant near-bed layer distribution (see Figure 3).

In Figure 4, the profiles of horizontal velocity for all 25 cells along the channel are plotted in one location. The profiles are shown for the original layering and the modified layering. The analytical solution is included as a reference. One can see that in the original situation, the velocity profiles are distorted in some cells, especially near the bottom. This occurs in the cells directly downstream of the point where the bottom crosses the layer interface. The velocity profiles obtained with the modified layering show much less variation and coincide quite well with the analytical velocity profile.

The bottom shear stress should be constant along the channel. In Figure 5 one can see that the large near-bed layer thickness ratio introduces a discontinuity in bottom shear stress. Using the proposed remapping, the variation is greatly reduced.

To test the method for use in more general flow situations, we implemented the approach in the z-layer module of the Delft3D modelling system (http://oss.delft3d.nl) and ran simulations of the flow over a bottom bump passing through a number of z-layers (applying again the k-ε turbulence model). Using the new approach, velocity profiles and bottom shear stress distributions were found to be much smoother than those obtained using the original layering. This provides good prospects for real-life applications using the proposed method. Results will be presented in a follow-up paper.

DISCUSSION

It should be noted that the proposed remapping causes neighbour cells to be shifted with respect to each other (see Figure 3). This should be taken into account in the computation of advection. It should however be noted that one already has to adequately account for advection effects at bottom steps, as done e.g. in Kleptsova et al. [6]. Otherwise, the solution may deteriorate due to spurious form drag. The combination of these two methods is therefore interesting to investigate. The layer remapping proposed by Kleptsova et al. [6] preserves an equidistant near-bed layering, so no conflicts are expected in the combination. The use of an Eulerian-Lagrangian advection scheme may also offer a solution here.

We also note that the boundary conditions applied in the k-ε turbulence model can have a significant influence on the results. We experimented with different ways of specifying the bottom boundary condition for k and ε and found that only a few methods provide accurate and stable results. Details on this subject are beyond the scope of this paper.

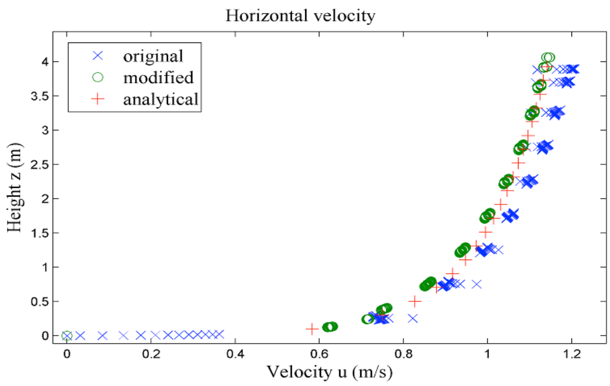

Figure 4. Velocity profiles for uniform channel flow; original layer distribution (blue crosses), modified near-bed layer distribution (green circles) and analytical solution (red plusses).

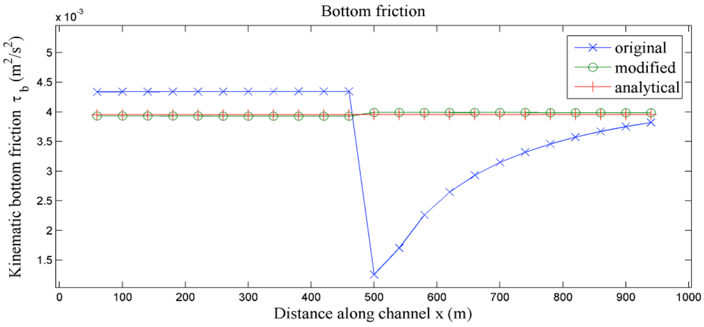

Figure 5. Shear stress variation along the channel for uniform channel flow; original layering (blue crosses), modified near-bed layering (green circles) and analytical solution (red plusses).

CONCLUSIONS

Accurate bottom shear stress computation for uniform channel flow in z-layer models can be achieved through a local remapping to an equidistant layering near the bottom. The approach functions both in combination with an algebraic mixing-length turbulence model and the k-ε turbulence model. Preliminary tests using the new method for the flow over a bottom bump show promising results. The simple modifications allow the direct use of computed bottom shear stresses for morphodynamical simulations.

REFERENCES

[1] Adcroft A., Hill C., Marshall J., "Representation of topography by shaved cells in a height coordinate ocean model", *Monthly Weather Review*, Vol. 125, No. 9, (1997), pp 2293-2315.

[2] Arya S.P., "Finite-difference errors in estimation of gradients in the atmospheric surface layer", *J. Appl. Meteor.*, Vol. 30, No. 2, (1991), pp 251-253.

[3] Beckmann A., Döscher R., "A method for improved representation of dense water spreading over topography in geopotential-coordinate models", *J. Phys. Oceanogr.*, Vol. 27, No. 4, (1997), pp 581-591.

[4] Chen X., "Using a piecewise linear bottom to fit the bed variation in a laterally averaged, z-co-ordinate hydrodynamic model", *Int. J. Numer. Methods Fluids*, Vol. 44, No. 11, (2004), pp 1185-1205.

[5] Kester J. van, Vatvani D. and Bijvelds M., "3D simulation of the hydrodynamics in lakes with Delft3D-FLOW", *Proc. of the 5th int. Conf. on Hydrodynamics*, Tainan, Taiwan, (2002), pp 1-11.

[6] Kleptsova O., Stelling G. and Pietrzak J., "An accurate momentum advection scheme for a Z-level coordinate models", *Ocean Dynamics*, Vol. 60, No. 6, (2010), pp 1-15.

[7] Nezu, I. and Nakagawa, H., "*Turbulence in Open-Channel Flows*", IAHR Monograph, Balkema, (1993).

[8] Pacanowski R.C. and Gnanadesikan A., "Transient response in a Z-level ocean model that resolves topography with partial cells", *Monthly Weather Review*, Vol. 126, No. 12, (1998), pp 3248-3270.

[9] Penduff T., Le Sommer J., Barnier B., Treguier A.-M., Molines J.-M. and Madec G., "Influence of numerical schemes on current-topography interactions in 1/4° global ocean simulations", *Ocean Science*, Vol. 3, No. 4, (2007), pp 509-524.

[10] Phillips, N.A., "A coordinate system having some special advantages for numerical forecasting", *J. Meteorol.*, Vol 14, No. 2, (1957), pp 184-185.

[11] Stelling G., "Compact differencing for stratified free surface flow", *Advances in Hydro-Science and Engineering*, Bejing, China, Vol. 2, (1995), pp 378-386.

[12] Stelling G. and Kester J. van, "On the approximation of horizontal gradients in sigma coordinates for bathymetry with steep bottom slope", *Int. J. Numer. Methods Fluids*, Vol. 18, No. 10, (1994), pp 915-935.

[13] Winton M., Hallberg R. and Gnanadesikan A., "Simulation of density-driven frictional downslope flow in Z-coordinate ocean models", *J. Phys. Oceanogr.*, Vol. 26, No. 11, (1998), pp 2163-2174.

[14] Zhao G., "*Numerical simulation of complex channel flows*", PhD thesis, Stanford University, Department of Civil and Environmental Engineering, (2007).

[15] Zijlema M., "*Een studie naar randvoorwaarden en discretisaties bij een vaste wand voor tubulentiemodellen in incompressibele stromingsmodellen*", MSc thesis, Delft University of Technology, Faculty of Civil Eng. and Geosciences, Dep. of Hydraulic Eng., (1991).